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Stability of Attractive Bose–Einstein Condensates

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We propose the critical nonlinear Schrödinger equation with a harmonic potential as a model of attractive Bose–Einstein condensates. By an elaborate mathematical analysis we show that a sharp stability threshold exists with respect to the number of condensate particles. The value of the threshold agrees with the existing experimental data. Moreover with this threshold we prove that a ground state of the condensate exists and is orbital stable. We also evaluate the minimum of the condensate energy.

KEY WORDS: Attractive Bose–Einstein condensates; nonlinear Schrödinger equation; stability; ground state; variational arguments.

1. INTRODUCTION

Experimental realization of Bose–Einstein condensation in ultracold vapors of ^{7}Li atoms[1] opened a new field in the study of macroscopic quantum phenomena.[6, 10, 11, 32] Bose–Einstein Condensates (BEC) with attractive interactions are known to be metastable in spatially localized systems, provided that the number of condensed particles, say N, is below a critical value N_c, while they are unstable if $N \geqslant N_c$.[6, 10, 11, 32] We call this number N_c the threshold of stability. In light of the experimental observation of attractive BEC, a remarkable series of theoretical researches are conducted[6, 10, 11, 32] in terms of the Gross–Pitaevskii (GP) equation.[9, 25] But, as we will show later, except in two dimensional space, the GP equation is not an accurate model for the attractive BEC since this equation does not give rise to a threshold of stability that is characteristic of attractive BEC. This leads to the following question: Can we modify the GP equation and establish a

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new theoretical framework for attractive BEC? Furthermore, will the new framework yield qualitative and quantitative results that agree with existing experimental observations?

We find that for the GP equation the break comes from the nonlinear interaction power term $|\phi|^2 \phi$. From GP equation, we propose a modified equation with a nonlinear interaction power term $|\phi|^p \phi$. By a scaling argument and known mathematical results, it is shown that $p = 4/D$ is a unique choice for an attractive BEC model, where D is the space dimension. And in this case, the corresponding modified equation is just the critical nonlinear Schrödinger equation with a harmonic potential. Then we can apply Weinstein's idea for classic critical nonlinear Schrödinger equation[33] and through an elaborate mathematical analysis we show that for the modified equation with the nonlinear interaction power term $|\phi|^{4/D} \phi$, a sharp stability threshold exists with respect to the number of condensate particles. Moreover the value of the threshold predicted by our model equation agrees with the existing experimental data.[1] With this threshold we also show that a ground state of the condensate exists and is orbital stable, which is another characteristic of attractive BEC. We also evaluate the minimum of the condensate energy by a variational computation.

Thus we claim that the critical nonlinear Schrödinger equation with a harmonic potential is a proper model equation of attractive BEC which realizes qualitative and quantitative results that agree with existing experimental observations. The two dimensional GP equation is consistent with our claim, but the three dimensional GP equation can not be regarded as a model of attractive BEC since in essence it is contradictory with the characteristics of attractive BEC. On the other hand, we remark that for repulsive BEC, GP equation is an accurate model (see ref. 6).

In the following we first discuss the GP equation and propose the modified model equation for attractive BEC. In Section 3, we state a rigorous local theory for the modified equation. In Section 4, we show the threshold of the stability to exist in the modified equation, which agrees with the experimental data. In Section 5, with the sharp threshold we further use a variational argument to prove the existence and orbital stability of the ground state in the condensate.

2. THE MODEL OF ATTRACTIVE BEC

At low enough temperature, neglecting the thermal and quantum fluctuations, a Bose condensate can be represented by a complex wave function $\tilde{\psi}(\tilde{x}, \tilde{t})$ that obeys the dynamics of the GP equation.[9, 25] Specifically, we consider a condensate of N particles of mass m and negative effective scattering length $\tilde{a}$ in a radial confining harmonic potential $V(r) = m\omega^2 r^2/2$.

As in ref. 10, using variables rescaled by the natural quantum harmonic oscillator units of time $\tau_0 = 1/\omega$ and length $L_0 = \sqrt{\hbar/(m\omega)}$: $t = \tilde{t}/(2\tau_0)$, $x = \tilde{x}/L_0$, $\tilde{\psi}(\tilde{x}, \tilde{t}) = \psi(x, t)$ and $a = 8\pi\tilde{a}/L_0$, we get the GP equation to describe the condensate as follows:

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + |x|^2\,\psi + a\,|\psi|^2\,\psi, \qquad t \geqslant 0, \quad x \in R^D \tag{2.1}$$

We note that in Eq. (2.1) $a < 0$ since $\tilde{a}$ is negative. Equation (2.1) is regarded as a model of attractive BEC (see refs. 6, 10, 11, 32 and the references therein). It is well known that Eq. (2.1) possesses the following two conserved quantities

$$A = \int_{R^D} (|\nabla\psi|^2 + |x|^2\,|\psi|^2 + \tfrac{1}{2}a\,|\psi|^4)\,dx \tag{2.2}$$

$$N = \int_{R^D} |\psi|^2\,dx \tag{2.3}$$

A and N represent the total energy and particle number in condensate respectively.

Now we explain why GP equation is not an accurate model for an attractive BEC. From an extended Zakharov's theory,[3, 31] one sees that the condensate wave function $\psi(x, t)$ collapses when the energy $A < 0$. Now we take N_0 to be sufficiently small and $\psi(x, 0)$ to be an arbitrarily given initial density such that $\int_{R^D} |\psi(x, 0)|^2\,dx = N_0$. For $\lambda > 0$, let $\psi_\lambda = \lambda^{D/2}\psi(\lambda x, 0)$. Then we still have $\int_{R^D} |\psi_\lambda|^2\,dx = N_0$. And from (2.2), the energy corresponding to ψ_λ is

$$A_\lambda = \int_{R^D} [\lambda^2\,|\nabla\psi(x, 0)|^2 + \lambda^{-2}\,|x|^2\,|\psi(x, 0)|^2 + \tfrac{1}{2}a\lambda^D\,|\psi(x, 0)|^4]\,dx \tag{2.4}$$

Thus for given $\psi(x, 0)$, when $D \geqslant 3$, we can always take λ large enough so that $A_\lambda < 0$ since $a < 0$. Hence, by the above mentioned extened Zakharov's theory, the solution of Eq. (2.1) with the initial density ψ_λ collapses. This means that if $D \geqslant 3$, collapse can occur, even if the particle number in condensate is sufficiently small. Therefore, there is no critical value of condensates at all. This is contradictory with the known properties of attractive BEC.[1] Thus the GP equation (2.1) with $D \geqslant 3$ is not an accurate model for attractive BEC.

We know that the collapse in essence comes from nonlinear interacting. From (2.4) we see that just as λ^D that comes from the nonlinear interaction energy $\int_{R^D} |\psi|^4\,dx$ in Eq. (2.1) can not balance λ^2 that comes from the kinetic energy $\int_{R^D} |\nabla\psi|^2\, dx$ in Eq. (2.1) when $D \geqslant 3$, the negative energy appears and the collapse occurs for sufficiently small particle number N. Thus set $p > 1$ we replace $|\psi|^2$ by $|\psi|^p$ in Eq. (2.1) for modifying the balance between the nonlinear interaction energy and the kinetic energy. Then we get the equation

$$i\,\frac{\partial\psi}{\partial t} = -\Delta\psi + |x|^2\,\psi + a\,|\psi|^{p-1}\,\psi, \qquad t \geqslant 0, \quad x \in R^D \tag{2.5}$$

Equation (2.5) possesses the following conservation quantity of energy on time t,

$$I = \int_{R^D} \left(|\nabla\psi|^2 + |x|^2\,|\psi|^2 + \frac{2}{p+1}\,a\,|\psi|^{p+1} \right) dx \tag{2.6}$$

and the conservation quantity of particle number N that is the same as (2.3). We still put $\psi_\lambda = \lambda^{D/2}\psi(\lambda x, 0)$ for $\lambda > 0$. Then for $p > 4/D$, the same instability results as above always hold, so there is no threshold of stability. On the other hand, when $p < 4/D$, earlier mathematical studies[22, 38] show that for any particle number N, the condensate is always metastable, which is again contradictory with the known features of attractive BEC. Hence the only remaining choice for the nonlinear interaction exponent in Eq. (2.5) is $p = 4/D$. It turns out that this choice of p does indeed give rise to a threshold phenomenon. Thus we propose the following equation as a more appropriate model for an attractive BEC:

$$i\,\frac{\partial\psi}{\partial t} = -\Delta\psi + |x|^2\,\psi + a\,|\psi|^{4/D}\,\psi, \qquad t \geqslant 0, \quad x \in R^D \tag{2.7}$$

which is the critical nonlinear Schrödinger equation with a harmonic potential. From a rigorous viewpoint of mathematics, in the following we first study the local well-posedness of Eq. (2.7).

3. LOCAL WELL-POSEDNESS

We impose the initial data of Eq. (2.7) as follows.

$$\psi(0, x) = \psi_0, \qquad x \in R^D \tag{3.1}$$

For Eq. (2.7), we define the energy space in the course of nature by

$$H := \left\{ \varphi \in H^1(R^D),\ \int |x|^2\,|\varphi|^2\,dx < \infty \right\} \tag{3.2}$$

Here and hereafter, for simplicity, we denote $\int_{R^D} \cdot\, dx$ by $\int \cdot\, dx$. H becomes a Hilbert space, continuously embedded in $H^1(R^D)$, when endowed with the inner product

$$\langle \varphi, \phi \rangle_H = \int \nabla\varphi\, \nabla\bar{\phi} + \varphi\bar{\phi} + |x|^2\, \varphi\bar{\phi}\, dx \tag{3.3}$$

whose associated norm we denote by $\|\cdot\|_H$.

We define the energy functional in H as follows.

$$E(\varphi) := \int |\nabla\varphi|^2 + |x|^2\, |\varphi|^2 + \frac{1}{1+2/D}\, a\, |\varphi|^{2+4/D}\, dx \tag{3.4}$$

In terms of the smoothness off the time 0 of Schrödinger kernel for potentials of quadratic growth provided by Fujiware,[8] Oh[22] established the local well-posedness of the Cauchy problem of Eq. (2.7) with initial data (3.1) in the corresponding energy space H (also see ref. 3).

Proposition 3.1. Let $\psi_0 \in H$. Then there exists a unique solution ψ of the Cauchy problem (2.7), (3.1) in $C([0, T); H)$ for some $T \in (0, \infty]$ (maximal existence time), and $\psi(t, \cdot)$ satisfies the following two conservation laws of particle number N:

$$\int |\psi|^2\, dx = \int |\psi_0|^2\, dx$$

and energy

$$E(\psi) = E(\psi_0)$$

for all $t \in [0, T)$. Furthermore we have the following alternatives: $T = \infty$ or else $T < \infty$ and $\lim_{t \to T} \|\psi\|_H = \infty$ (collapse).

By a direct calculation (also see ref. 3) we have

Proposition 3.2. Let $\psi_0 \in H$ and ψ be a solution of the Cauchy problem (2.7), (3.1) in $C([0, T); H)$. Put $J(t) := \int |x|^2\, |\psi|^2\, dx$. Then one has $J'(t) = -2\Im \int x\psi\, \nabla\bar{\psi}\, dx$ and

$$J''(t) = 8E(\psi_0) - 16 \int |x|^2\, |\psi|^2\, dx \tag{3.5}$$

Thus one can imply that

Corollary 3.1. Let $\psi_0 \in H$. Then when $E(\psi_0) < 0$, the solution ψ of the Cauchy problem (2.7), (3.1) collapse in a finite time. In other words, there is $T < \infty$ such that

$$\lim_{t \to T} \|\psi\|_H = \infty$$

Remark 3.1. From Proposition 3.2 we can get that when $E(\psi_0) \geqslant 0$, that is for zero energy or positive energy, there are also collapse solutions of the Cauchy problem (2.7), (3.1). In ref. 37, By constructing a kind of cross-constrained minimization problem we get a sharp threshold of collapse solutions of the Cauchy problem (2.7), (3.1) in terms of the cross-invariant sets.

Remark 3.2. Consider Eq. (2.5). From ref. 22, when $1 < p < 1 + 4/D$, we have the global well-posedness of the Cauchy problem (2.5), (3.1). And from ref. 3, when $p \geqslant 1 + 4/D$, there are collapse solutions to exist for the Cauchy problem (2.5), (3.1). So we call the nonlinearity $|\psi|^{4/D}\psi$ critical nonlinearity.

Remark 3.3. In Eq. (2.5), we replace $|x|^2$ by a general real valued potential $V(x)$. Then we get the following nonlinear Schrödinger equation with a potential $V(x)$:

$$i\,\partial\psi/\partial t = -\Delta\psi + V(x)\,\psi + a\,|\psi|^{p-1}\,\psi, \qquad t \geqslant 0, \quad x \in R^D \tag{3.6}$$

Since Yajima[35] showed that for super-quadratic potentials, the Schrödinger kernel is nowhere C^1, from Oh[22] it is known that quadratic potentials are the highest order potential for local well-posedness of Eq. (3.6). Then $V(x) = |x|^2$ in potentials is critical for the local existence of the Cauchy problem (3.6), (3.1). At the same time, from Remark 3.2, $p = 1 + 4/D$ in dimension D is also the critical value for the global existence of the Cauchy problem (3.6), (3.1). Thus we get that Eq. (2.7) is critical both in potentials for the local existence and in dimension for the global existence.

4. SHARP THRESHOLD OF THE STABILITY

In this section, we use the variational approach to establish the relation between a classic elliptic equation and Eq. (2.7). Then we can get the sharp threshold of the stability for Eq. (2.7).

Consider the nonlinear scalar field equation

$$-\Delta u+\frac{2}{D}u-|u|^{4/D}u=0, \qquad u\in H^1(R^D) \tag{4.1}$$

From refs. 33 and 12, we have the following lemma.

Lemma 4.1. Equation (4.1) has a unique positive radially symmetric solution $Q(x)$, that is, a solution $Q(x)$ depending only on $|x|$. Moreover $(D/(2+D))(\int Q^2\,dx)^{2/D}$ is the minimum of the functional

$$I(\psi)=\left(\int|\nabla\psi|^2\,dx\right)\left(\int|\psi|^2\,dx\right)^{2/D}\Big/\left(\int|\psi|^{2+4/D}\,dx\right), \qquad \phi\in H \tag{4.2}$$

Remark 4.1. From Lemma 4.1, we can get

$$\int|\psi|^{2+4/D}\,dx\leqslant\frac{2+D}{D}\left(\int Q^2\,dx\right)^{-2/D}\left(\int|\nabla\psi|^2\,dx\right)\left(\int|\psi|^2\,dx\right)^{2/D} \tag{4.3}$$

which is just the Gagliardo–Nirenberg inequality. Moreover from Eq. (4.1), $Q(x)$ satisfies that

$$(1+2/D)\int|\nabla Q|^2\,dx=\int|Q|^{2+4/D}\,dx \tag{4.4}$$

Next we state an inequality as follows (also see refs. 30 and 33).

Lemma 4.2. Let $u\in H$. Then we have

$$\int|u|^2\,dx\leqslant\frac{2}{D}\left(\int|\nabla u|^2\,dx\right)^{1/2}\left(\int|x|^2\,|u|^2\,dx\right)^{1/2}$$

Proof. From the identity

$$-D\int|u|^2\,dx=2\Re\int\bar{u}x\cdot\nabla u\,dx$$

and the Cauchy–Schwarz inequality, it follows the above inequality.

We also note that "$2/D$" is the best constant for the above inequality with equality holding for the function $u=\exp(-\frac{1}{2}|x|^2)$.

Then we give a lemma about the collapse of the solutions of Eq. (2.7).

Lemma 4.3. If $\psi_0 \not\equiv 0$ satisfies that

$$J(0) = \int (|x|^2 \, |\psi_0|^2) \, dx \geqslant E(\psi_0)$$

then the solutions ψ of Eq. (2.7) with the initial data (3.1) collapse in a finite time.

Proof. From Proposition 3.2 we have

$$J(t) = \beta \sin(4t + \theta) + \tfrac{1}{2} E(\psi_0) \tag{4.5}$$

where β and θ are constants determined by $J(0)$ and $J'(0)$. Moreover

$$\beta^2 = [J(0) - \tfrac{1}{2} E]^2 + \tfrac{1}{16} [J'(0)]^2 \tag{4.6}$$

Thus if $J(0) \geqslant E$, (4.5) and (4.6) imply that there exists $T < \infty$ such that

$$\lim_{t \to T} J(t) = 0$$

By Lemma 4.2 we get that

$$\lim_{t \to T} \int |\nabla \psi|^2 \, dx = \infty$$

This shows that $\psi(x, t)$ collapses.

Now we can claim the sharp threshold of the stability for Eq. (2.7).

Theorem 4.1. Let $Q(x)$ be the positive radially symmetric solution of Eq. (4.1). If ψ_0 satisfies $\psi_0 \in H$ and $\|\psi_0\|_{L^2} < |a|^{-D/4} \, \|Q\|_{L^2}$, then the solution $\psi(t, x)$ of the Cauchy problem (2.7), (3.1) exists globally in time. At the same time, for arbitrary positive λ and complex c satisfying $|c| \geqslant 1$, if we take initial data $\psi_0 \in H$ such that $\psi(x, 0) = c\lambda^{D/2} \, |a|^{-D/4} \, Q(\lambda x)$, then $\|\psi_0\|_{L^2} = |c| \, |a|^{-D/4} \, (\int Q^2 \, dx)^{1/2} \geqslant |a|^{-D/4} \, \|Q\|_{L^2}$ and the solutions $\psi(t, x)$ of the Cauchy problem (2.7), (3.1) collapse in a finite time.

Proof. Let $\psi(t, x) \in C([0, T); H)$ be a solution of the Cauchy problem (2.7), (3.1). From (3.4) and Proposition 3.1, applying Lemma 4.1 we get that

$$\int \left\{ \left[1 + a \left(\frac{\int |\psi|^2 \, dx}{\int Q^2 \, dx} \right)^{2/D} \right] |\nabla \psi|^2 + |x|^2 \, |\psi|^2 \right\} dx \leqslant E(\psi_0) \tag{4.7}$$

Thus from

$$\int |\psi|^2 \, dx < |a|^{-D/2} \int Q^2 \, dx$$

we imply that $\int |\nabla\psi|^2 \, dx$ and $\int (|x|^2 \, |\psi|^2) \, dx$ are bounded for $t \in [0, T)$ and any $T < \infty$. By Proposition 3.1, it yields that $\psi(t, x)$ globally exists in $t \in [0, \infty)$.

Now we take the initial data such that

$$\psi(x, 0) = c\lambda^{D/2} \, |a|^{-D/4} \, Q(\lambda x)$$

with arbitrary positive λ and complex c satisfying $|c| \geqslant 1$. Then

$$\int |\psi(x, 0)|^2 \, dx = |c|^2 \, |a|^{-D/2} \int Q^2 \, dx \geqslant |a|^{-D/2} \int Q^2 \, dx$$

And from (3.4), (4.4), the corresponding energy is

$$E = (1 - |c|^{4/D}) \, |c|^2 \, |a|^{-D/2} \, \lambda^2 \int |\nabla Q|^2 \, dx + J(0) \leqslant J(0)$$

Thus Lemma 4.3 yields that $\psi(t, x)$ collapses in a finite time.

According to Theorem 4.1 we thus claim that

$$N_c = [\, |a|^{-D/4} \, \|Q\|_{L^2}]^2 = |a|^{-D/2} \int Q^2 \, dx \tag{4.8}$$

is the critical value of condensed particles, which is just the sharp threshold for the stability of Eq. (2.7).

From the viewpoint of Mathematics, Theorem 4.1 is surprise because the critical value in initial data for global existence of Eq. (2.7) is $|a|^{-D/4} \, \|Q\|_{L^2}$, which is just the critical value in initial data for global existence of the classic critical nonlinear Schrödinger equation without any potential (see ref. 33)

$$i \, \frac{\partial\psi}{\partial t} = -\Delta\psi + a \, |\psi|^{4/D} \, \psi, \qquad t \geqslant 0, \quad x \in R^D \tag{4.9}$$

Since the minimal blowup solutions of both Eq. (4.9) and Eq. (2.7) are generated by the same positive radially symmetric solution of Eq. (4.1), many mature results on the blowup properties of Eq. (4.9) (see refs. 15, 16, and 18) can be used in Eq. (2.7).

From the viewpoint of Physics, Theorem 4.1 is more meaningful since the existence of critical value $|a|^{-D/4}\,\|Q\|_{L^2}$ realizes the properties of attractive BEC by the model equation (2.7). In Section 2 we have claimed that Eq. (2.7) is a unique selection as a model equation of attractive BEC. Theorem 4.1 justified that this selection is accurate by a rigorous mathematical statement. Furthermore we can use the experimental data in ref. 1 to verify the accuracy of the above statement. From refs. 1 and 10, we get that $a=-1.148\times 10^{-2}$. We take $D=2$ and from ref. 33, $\int Q^2\,dx=2\pi\times 1.862....$ Thus we get the critical value N_c of attractive BEC is

$$N_c=[\,|a|^{-D/4}\,\|Q\|_{L^2}]^2=(1.148\times 10^{-2})^{-1}\times 2\pi\times 1.862...=1019 \quad (4.10)$$

which is well consistent with the experimental results in ref. 1.

5. THE STABLE GROUND STATES

In this section, we study the stable ground state of Eq. (2.7). From refs. 33 and 34, when $\|\psi(0,\cdot)\|_{L^2}<|a|^{-D/4}\,\|Q\|_{L^2}$, Eq. (4.9) has no any standing waves to exist. When $\|\psi(0,\cdot)\|_{L^2}=|a|^{-D/4}\,\|Q\|_{L^2}$, there are standing waves with ground state in Eq. (4.9). But these standing waves are unstable with collapse. For Eq. (2.7), the situation is completely different. Firstly we have the following lemma (also see ref. 38).

Lemma 5.1. Let $1\leqslant q<(D+2)/(D-2)$ when $D\geqslant 3$ and $1\leqslant q<\infty$ when $D=1, 2$. Then the embedding $H\hookrightarrow L^{q+1}(R^D)$ is compact.

Proof. We firstly show it for $q=1$.

Since $H\hookrightarrow H^1(R^D)$ continuously, it follows that by Sobolev's embedding theorem $H\hookrightarrow L^{q+1}(R^D)$ continuously. Now let $(u_n)_n\subset H$ be a sequence such that

$$u_n\rightharpoonup 0, \qquad \text{weakly in } H$$

Then we have

$$u_n\rightharpoonup 0, \qquad \text{weakly in } H^1(R^D) \quad (5.1)$$

Moreover we have $M:=\sup_n \|u_n\|_H<\infty$. Let $\varepsilon>0$, then there exists $B>0$ such that $1/|x|^2\leqslant\varepsilon$ for $|x|\geqslant B$. For B, from (5.1) we have

$$u_n\to 0 \qquad \text{in } L^2(\{\,|x|\leqslant B\}\,)$$

It follows that there exists m such that

$$\int_{|x|\leqslant B}|u_n|^2\,dx\leqslant\varepsilon \qquad \text{for} \quad n\geqslant m$$

Then when $n \geqslant m$, we get

$$\int |u_n|^2\,dx = \int_{|x| \leqslant B} |u_n|^2\,dx + \int_{|x| \geqslant B} |u_n|^2\,dx$$

$$\leqslant \varepsilon + \varepsilon \int_{|x| \geqslant B} |x|^2\,|u_n|^2\,dx \leqslant \varepsilon + \varepsilon c M^2$$

Here and hereafter c denotes various positive constants. Thus we get that

$$u_n \to 0 \qquad \text{in } L^2(R^D)$$

It follows that $H \hookrightarrow L^2(R^D)$ is compact.

For $q > 1$, using the conclusion of $q = 1$ and the Gagliardo–Nirenberg inequality,

$$\|u\|^{q+1}_{L^{q+1}(R^D)} \leqslant c\,\|\nabla u\|^{(D/2)(q-1)}_{L^2(R^D)}\,\|u\|^{q+1-(D/2)(q-1)}_{L^2(R^D)}$$

we can get the conclusion immediately.

Now for $N > 0$, we define a variational problem as follows.

$$d_N := \inf_{\{u \in H,\, \int |u|^2\,dx = N\}} E(u) \tag{5.2}$$

Lemma 5.2. If N in (5.2) satisfies, $N^{1/2} < |a|^{-D/4}\,\|Q\|_{L^2}$, then we have

$$d_N = \min_{\{u \in H,\, \int |u|^2\,dx = N\}} E(u) \tag{5.3}$$

Proof. Let $u_n \in H$ such that

$$\int |u_n|^2\,dx \to N, \qquad E(u_n) \to d_N \tag{5.4}$$

Since

$$N^{1/2} < |a|^{-D/4}\,\|Q\|_{L^2} \tag{5.5}$$

(5.4) implies that there exists m such that for $n > m$ one has

$$\int |u_n|^2\,dx < |a|^{-D/2}\,\|Q\|^2_{L^2} \tag{5.6}$$

$$E(u_n) < d_N + 1 \tag{5.7}$$

By (3.4) and Lemma 4.1, it follows from (5.7) that

$$\int\left\{\left[1+a\left(\frac{\int |u_n|^2\,dx}{\int Q^2\,dx}\right)^{2/D}\right]|\nabla u_n|^2+|x|^2\,|u_n|^2\right\}dx\leqslant d_N+1 \tag{5.8}$$

Thus from (5.6) we imply that $\int |\nabla u_n|^2\,dx$ and $\int (|x|^2\,|u_n^2|)\,dx$ are bounded for all $n>m$. It yields that $\{u_n, n\in Z\}$ is bounded in H. Therefore there exists $u\in H$ such that

$$u_n \rightharpoonup u \qquad \text{in } H \tag{5.9}$$

By Lemma 5.1, then

$$u_n \to u \qquad \text{in } L^2(R^D) \tag{5.10}$$

$$u_n \to u \qquad \text{in } L^{2+4/D}(R^D) \tag{5.11}$$

Thus (5.10) implies that

$$\int |u|^2\,dx=N \tag{5.12}$$

From (5.9), (5.11) and (5.12), we get that $E(u)=d_N$. So (5.3) is true.

We denote the set of the minimizers of the minimization problem (5.3) by S_N. Then for any $u\in S_N$, there exists a Lagrange multiplier $\Lambda\in R$ such that u is a solution of the elliptic equation

$$-\Delta u+|x|^2\,u+\Lambda u+au\,|u|^{4/D}=0 \tag{5.13}$$

It follows that $\psi(t,x)=e^{i\Lambda t}u$ is a standing wave solution of Eq. (2.7), which is also called ground state since u is a minimizer of (5.3). Thus $e^{i\Lambda t}u(\cdot)$ is the orbit of u. It is obvious that for any $t\geqslant 0$, if u is a solution of (5.3), then $e^{i\Lambda t}u$ is also a solution of (5.3), that is $e^{i\Lambda t}u\in S_N$. Now in terms of Cazenave and Lions' argument,[5] we have the following orbital stability.

Lemma 5.3. Let N satisfy that $N^{1/2}<|a|^{-D/4}\,\|Q\|_{L^2}$. Then for arbitrary $\varepsilon>0$, there exists $\sigma>0$ such that for any $\psi_0\in H$, if

$$\inf_{u\in S_N}\|\psi_0-u\|_H<\sigma$$

then the solution $\psi(t,x)$ of the Cauchy problem (2.7)–(3.1) satisfies

$$\inf_{u\in S_N}\|\psi(t,\cdot)-u(\cdot)\|_H<\varepsilon, \qquad \text{for all} \quad t\geqslant 0$$

Proof. Firstly for any $\psi_0 \in H$ satisfying $\|\psi_0\|_{L^2} < |a|^{-D/4} \|Q\|_{L^2}$, from Theorem 4.1, the corresponding solution $\psi(t, x)$ of the Cauchy problem (2.7)–(3.1) is global and bounded in H. Now arguing by contradiction, if the conclusion of Lemma 5.3 does not hold, then there exist $\varepsilon > 0$, a sequence $(\psi_0^n)_{n \in Z}$ such that

$$\inf_{u \in S_N} \|\psi_0^n - u\|_H < \frac{1}{n} \tag{5.14}$$

and a sequence $(t_n)_{n \in Z}$ such that

$$\inf_{u \in S_N} \|\psi_n(t_n, \cdot) - u(\cdot)\|_H \geqslant \varepsilon \tag{5.15}$$

where ψ_n denotes the solution of the Cauchy problem (2.7)–(3.1) with initial datum ψ_0^n. From (5.14), we have

$$\int |\psi_0^n|^2 \, dx \to \int |u|^2 \, dx = N \tag{5.16}$$

$$E(\psi_0^n) \to E(u) = d_N \tag{5.17}$$

It follows from (5.16), (5.17) and the conservation laws in Proposition 3.1 that $(\psi_n(t_n, \cdot))_{n \in Z}$ is a minimizing sequence for the problem (5.3). Therefore there exists $u \in S_N$ such that

$$\|\psi_n(t_n, \cdot) - u\|_H \to 0 \qquad \text{as} \quad n \to \infty$$

This is contradictory with (5.15). Thus this proves Lemma 5.3.

By Lemma 5.2 and Lemma 5.3, thus we can claim the following theorem.

Theorem 5.1. Let N satisfy that $N^{1/2} = \|\psi(0, \cdot)\|_{L^2} < |a|^{-D/4} \|Q\|_{L^2}$. Then the minimization problem (5.3) has minimizers to exist. And Eq. (2.7) has standing waves with ground state. Moreover these standing waves are orbitally stable.

Thus we see that although the global existence and the collapse properties of Eq. (4.9) are consistent with the corresponding properties of Eq. (2.7), on the standing waves, Eq. (4.9) is essentially different from Eq. (2.7). This difference just comes from Eq. (2.7) having a harmonic potential. Moreover experimental attractive BEC has shown that below the critical value N_c of the condensed particles, the attractive BEC has metastable ground state to exist. Thus the conclusion of Theorem 5.1 is well consistent with the properties of attractive BEC.

From (5.13), one has

$$\int (|\nabla u|^2 + |x|^2 \, |u|^2 + \Lambda \, |u|^2 + a \, |u|^{2+4/D}) \, dx = 0 \tag{5.19}$$

On the other hand from (5.13) we also have Pohzaev identity,

$$\int \left[\Lambda \, |u|^2 + (1 - 2/D) \, |\nabla u|^2 + (1 + 2/D) \, |x|^2 \, |u|^2 + \frac{D}{D+2} \, a \, |u|^{2+4/D} \right] dx \tag{5.20}$$

From (5.19) and (5.20) we have

$$\int \left(|\nabla u|^2 - |x|^2 \, |u|^2 + \frac{D}{2+D} \, a \, |u|^{2+4/D} \right) dx = 0 \tag{5.21}$$

Thus by (3.4), we get the energy of the ground state is

$$E_N = 2 \int (|x|^2 \, |u|^2) \, dx \tag{5.22}$$

where u is the minimizer of (5.3) subject to $N < N_c$. So (5.22) is also the minimal energy of the condensate subject to $N < N_c$.

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